НАЦИОНАЛЬНЫЙ ИССЛЕДОВАТЕЛЬСКИЙ УНИВЕРСИТЕТ

Nash implementability and related properties of the union of externally stable sets

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Alternatives, preferences, choices

A – the *general set* of alternatives.

X – the *feasible set* of alternatives: $X \subseteq A \wedge X \neq \varnothing$.

P – *strict social preferences*, $P \subseteq A^2$, $(x, y) \in P \Rightarrow (y, x) \notin P$.

P is presumed to be complete: $\forall\, x, y \in A, x \neq y \Rightarrow ((x, y) \in P \vee (y, x) \in P)$.

A *preference-based choice correspondence* is a mapping S: $2^A \backslash \varnothing \times 2^{A \times A} \rightarrow 2^A$

with arguments X and P and values in the set of subsets of X.

It is presumed that S depends on X and P only through restriction of P on X:

$$S = S(X, P) = S(P|_X) \subseteq X$$

i.e. choices are dependent on preferences for available alternatives only.

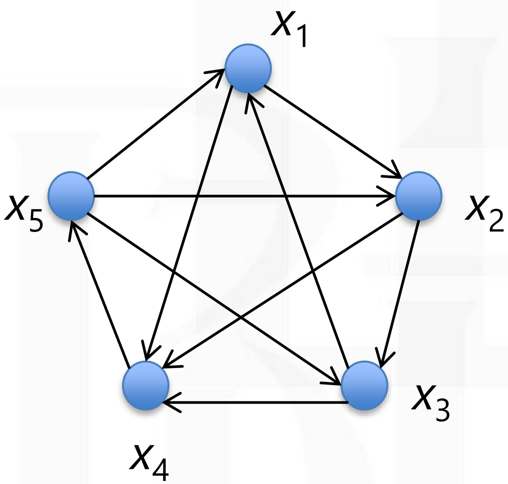
Tournament solutions

A *tournament solution* is a social preference based-choice correspondence *S* that has the following properties:

1. *Nonemptiness*: $\forall X, \forall P, \; S(P|_X) \neq \varnothing$;
2. *Neutrality*: permutation of alternatives' names and social choice commute;
3. *Condorcet consistency*: if there is the Condorcet winner *w* for $P|_X$ then $S(P|_X) = \{w\}$.

	x_1	x_2	x_3	x_4	x_5
x_1	0	1	0	1	0
x_2	0	0	1	1	0
x_3	1	0	0	1	0
x_4	0	0	0	0	1
x_5	1	1	1	0	0

Tournament matrix

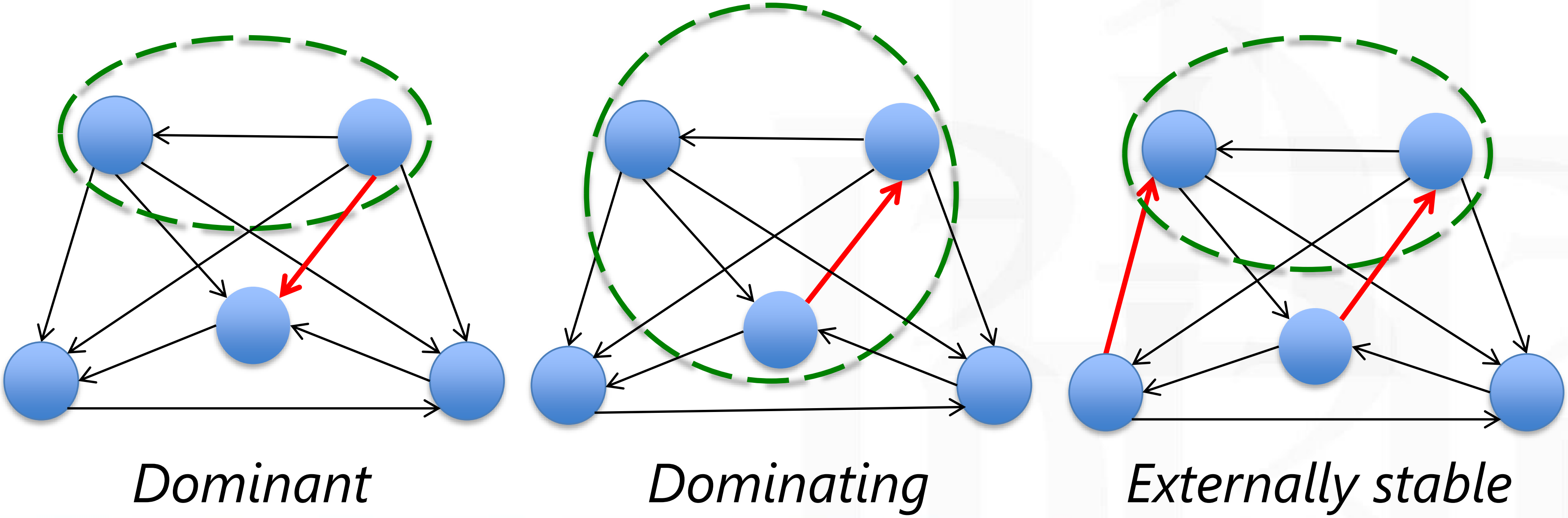


Tournament digraph

Stable sets

A nonempty subset *Y* of *X* is called

Dominant	if	$\forall x \in X \setminus Y,$	$\forall y \in Y: yPx$
Dominating	if	$\forall x \in X,$	$\exists y \in Y: yPx$
Externally stable	if	$\forall x \in X \setminus Y,$	$\exists y \in Y: yPx$



Minimal stable sets

A set *A* is called *minimal* with respect to a given property if *A* has the property and none of *A*'s proper nonempty subsets does.

Tournament solutions:

the union of all minimal

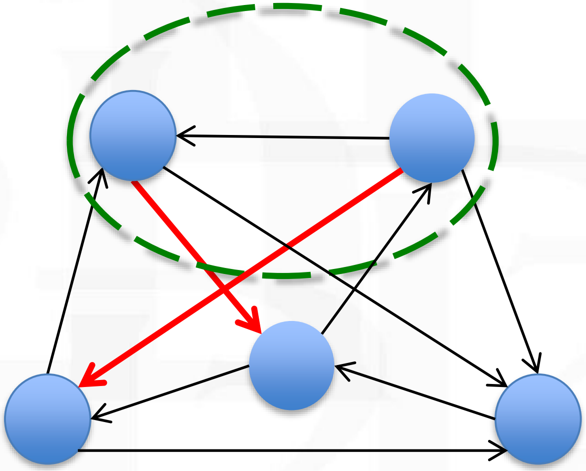
dominant sets	*TC*	(there is just one, a.k.a. the *top cycle*)
dominating sets	*D*	(Duggan 2013, Subochev 2016)
externally stable sets	*ES*	(Wuffl, Feld, Owen & Grofman 1989, Aleskerov & Kurbanov 1999, Subochev 2008)

Cooperative game interpretation

Sets of alternatives can be interpreted as *coalitions* (e.g. sport teams, political cliques etc.). External stability guaranties a victory of a coalition (represented by its champion) in a duel with any outsider (the "Three Musketeers" principle). Consequently, *ES* can be viewed as a solution of the following simple cooperative game:

- X is the set of players;
- Value function $v(Y)=1$ if Y is externally stable, $v(Y)=0$ otherwise.

Then *ES* is the support of Banzhaf and Shapley–Shubik power indices.



Externally stable

Properties

- *(generalized) Nash independence of irrelevant alternatives*:
  $$\forall X \subseteq A,\ \forall Y \subseteq X,\ S(X) \cap Y \neq \varnothing \ \Rightarrow S(Y) = S(X) \cap Y.$$
- *Weak Nash independence of irrelevant alternatives*:
  $$\forall X \subseteq A,\ \forall Y \subseteq X,\ S(X) \subseteq Y \ \Rightarrow S(Y) = S(X).$$
- *P-monotonicity (monotonicity w.r.t. social preferences)*:

$$\forall P_1, P_2 \subseteq A^2, \forall X \subseteq A, \forall a \in S(P_1|_X),\ (P_1|_{X \setminus \{a\}} = P_2|_{X \setminus \{a\}} \wedge \forall b \in X,\ aP_1 b \Rightarrow aP_2 b) \Rightarrow a \in S(P_2|_X).$$

- *Independence of social preferences for irrelevant alternatives*:

$$\forall P_1, P_2 \subseteq A^2,\ \forall X \subseteq A,\ (\forall a \in S(P_1|_X),\ \forall b \in X,\ aP_1 b \Leftrightarrow aP_2 b) \Rightarrow S(P_1|_X) = S(P_2|_X).$$

Theorem 1:

D does not satisfy any axiom from the list.

ES satisfies all listed axioms except NIIA.

The society and the majority rule

The ***society*** is a group G of n individual decision-makers (voters, experts etc.), $n>1$.

Each member of G has preferences for alternatives from A: $P_k \subseteq A^2$, $k \in G$.

$\mathbf{P} = \{P_k \subseteq A^2 \mid k \in G\}$ - profile of individual preferences.

We suppose that all possible P_k are linear orders.

A ***social choice correspondence*** is a mapping SC: $2^A \backslash \varnothing \times (2^{A \times A})^n \to 2^A$

with arguments X and **P** and values in the set of subsets of X.

We consider only those SC that depends on X and **P** only through restriction of **P** on X.

Social preferences is a mapping P: $(2^{A \times A})^n \to 2^{A \times A}$ with argument **P** and values in the set of all binary relations on A.

A special case of P – ***the majority rule*:**

$$xPy \Leftrightarrow |G_1|>|G_2|, \text{ where } G_1=\{k \in G \mid a\, P_k\, b\},\ G_2=\{k \in G \mid b\, P_k\, a\}.$$

Implementability and Maskin monotonicity

A social choice correspondence $S(\mathbf{P}|_X)$ is ***Nash implementable*** if for any feasible choice set X there is a non-cooperative game form Γ with a set of players G and set of outcomes X such that for any admissible profile $\mathbf{\Pi}$ the set of social choices coincides with the set of outcomes corresponding to Nash equilibria of the game $(\Gamma, \mathbf{P}|_X)$.

A social choice correspondence $S(\mathbf{P}|_A)$ is ***Maskin monotonic*** if for any feasible choice set X and any two admissible profiles **P** and **P*** the following holds:

$$\forall a \in S(\mathbf{P}|_X), (\forall b \in X, \forall k \in G, aP_k b \Rightarrow aP^*_k b) \Rightarrow a \in S(\mathbf{P^*}|_A)$$

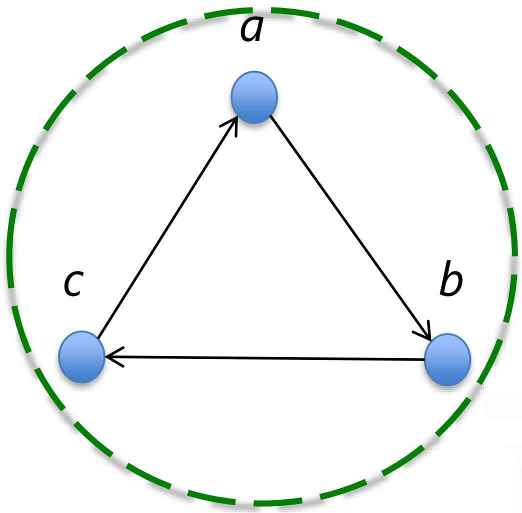
Maskin's theorem: S is Nash implementable only if it is Maskin monotonic.

Maskin monotonicity is almost sufficient for Nash implementability of S.

(Non)implementability of tournament solutions

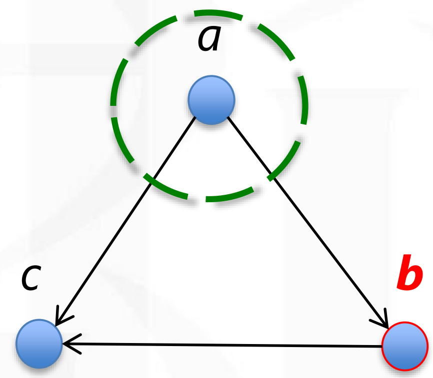
Condorcet consistency is incompatible with Maskin monotonicity.

P_1	P_2	P_3
a	*b*	*c*
b	*c*	*a*
c	*a*	*b*



Profile **P**

P_1	P_2	P_3
a	*b*	***a***
b	*c*	***c***
c	*a*	*b*



Profile **P***

If social preferences *P* are based on majority rule and if any set of *n* linear orders is admissible as a profile then no tournament solution is Maskin monotonic and, consequently, Nash implementable in a standard setting.

Implementation of a tournament solution

I.Özkal-Sanver and R.Sanver (2006, 2009) demonstrate that it is possible to Nash implement some tournaments solutions by set-valued hyperfunctions, when individual preferences are coherently extended over sets of alternatives.

A tournament solution *S* is ***Sanver monotonic*** if for any feasible choice set *X* and any two social preference relations *P* and *P** the following statement holds:

$$(\forall a \in S(P|_X), \forall b \in X, aPb \Rightarrow aP^*b) \Rightarrow S(P|_X) \subseteq S(P^*|_X)$$

Sanvers' theorem: Suppose

1) social preferences *P* are based on the majority rule;

2) *P* is a *tournament*, i.e. *P* is complete: $\forall\ x \neq y,\ xPy \vee yPx$;

3) individual preferences P_k are coherently extended over sets of alternatives,

then a tournament solution *S* is Nash implementable if it is Sanver monotonic.

Implementation of a tournament solution

The top cycle, the ultimate uncovered set, the minimal covering set, the bipartisan set are Sanver monotonic,

while the uncovered set, the Banks set, the Copeland set, the Slater set are not (Özkal-Sanver and Sanver 2009).

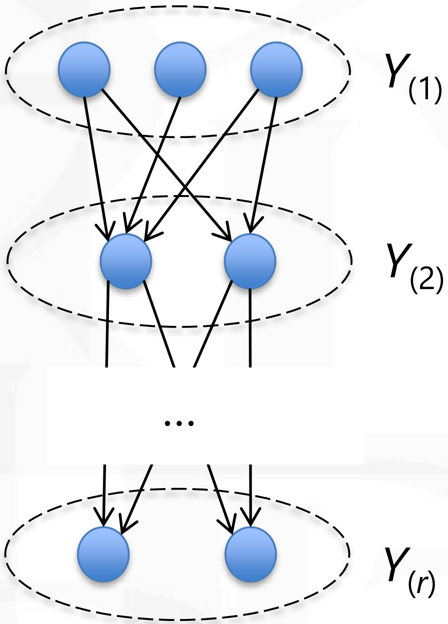
Theorem 2: *ES* is Sanver monotonic and therefore Nash implementable by a hyperfunction. *D* is not Sanver monotonic.

Ranking based on a tournament solution

Suppose, we are interested in ranking alternatives from X.

Then we may use the following procedure:

- Tournament solution $S(P, X)$ choses the set $Y_{(1)}$ of the best alternatives in X, $Y_{(1)}=S(P, X)$.
- Exclude these alternatives from X and apply S to the rest. $Y_{(2)}=S(P, X\backslash Y_{(1)})=S(P, X\backslash S(P, X))$ will be the set of the second-best alternatives in X.
- By repeated exclusion of the best alternatives determined at each step of the procedure the set X is separated into groups $Y_{(r)}=S(P, X\backslash(Y_{(r-1)}\cup Y_{(r-2)}\cup\ldots\cup Y_{(2)}\cup Y_{(1)}))$, and that is the ranking.



The properties of the ranking rule based on *ES*

Let $Q = Q(P|_X) \subseteq X^2$ denote this ranking of alternatives from X.

- ***Weak Pareto principle***: if a Pareto dominates b, then aQb.

The strong Pareto principle is violated.

- ***Weak monotonicity w.r.t the individual preferences*** (*Smith's monotonicity*):

$(\mathbf{P}|_{X\setminus\{a\}} = \mathbf{P^*}|_{X\setminus\{a\}} \wedge \forall k \in G, \forall b \in X, aP_k b \Rightarrow aP^*_k b) \Rightarrow (\forall b \in A, aQ(P|_X)b \Rightarrow aQ(P^*|_X)b)$.

- ***Independence of irrelevant classes of alternatives***

This is a weak form of the Arrow independence of irrelevant alternatives.

It is satisfied because *ES* satisfies Nash independence of irrelevant alternatives.

The covering relation and the uncovered set

The covering relation (Fishburn, 1977; Miller, 1980)

The covering relation $C(P|_X) \subseteq X^2$, is a strengthening of the strict social preferences P:

The covering relation C: $aCb \Leftrightarrow (aPb \wedge \forall c \in X, bPc \Rightarrow aPc)$.

N.B. $C(P|_X)$ is not a restriction of $C(P)$ on X: $C(P|_X) \not\equiv C(P) \cap X^2$!

The set of all alternatives that are not covered in X by any alternative is called ***the uncovered set*** of a feasible set X.

The uncovered set and the stable sets

Theorem 3: Suppose $|X|<\infty$. $a \in ES \Leftrightarrow \exists\, b \in UC: aPb \vee a \in UC$.

Corollary 1: *ES* is a union of the upper sections (w.r.t. *P*) of all uncovered alternatives and the uncovered set *UC* itself.

$$UC \subseteq ES$$

Theorem 4: Suppose $|X|<\infty$. $a \in D \Leftrightarrow \exists\, b \in UC: aPb$.

Corollary 2: *D* is a union of the upper sections (w.r.t. *P*) of all uncovered alternatives.

Corollary 3:
There is a polynomial algorithm for computing *ES* and *D*.

The non-emptiness result for infinite feasible sets

Proposition: Assume $R(a)$ is compact for all $a \in X$ then $UC \neq \varnothing$.
(Banks, Duggan & Le Breton, 2006)

Let $\Omega = (X, \{\omega\})$ be the topology generated by $\{P^{-1}(a) \mid a \in X\}$.

Theorem 5: Suppose X is compact in Ω. Then Theorem 4 holds.

That is, $a \in D \Leftrightarrow \exists\, b \in UC: aPb$.

Corollary: Suppose X is compact in Ω. Then $UC \neq \varnothing$ and either $D \neq \varnothing$ (by Theorem 5) or there is a Condorcet winner.
In both cases $ES \neq \varnothing$.

Publications

1. Subochev A., Aleskerov F., Pislyakov V. 2018. Ranking journals using social choice theory methods: A novel approach in bibliometrics. *Journal of Informetrics*, V. 12, Iss. 2, P. 416–429.

2. Subochev A. 2017. Axiomatic analysis of the union of minimal externally stable sets concerning application and implementation of this novel social choice correspondence. *Proceedings of the XVII April International Academic Conference on Economic and Social Development: in 4 volumes*. *Volume 1* (pp. 111-120). Moscow. National Research University Higher School of Economics. (in Russian)

3. Aleskerov F., Subochev A. 2013. Modeling optimal social choice: matrix-vector representation of various solution concepts based on majority rule. *Journal of Global Optimization*, V. 56, Iss. 2., P. 737-756.

4. Subochev, A. 2008. Dominant, Weakly Stable, Uncovered Sets: Properties and Extensions. Working paper WP7/2008/03. Moscow: SU – Higher School of Economics.

Thank you!

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