New moduli components of rank 2 bundles on projective space

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This probably happened due to the fact that the questions of irreducibility (solved by [T] in 2012-13), and smoothness (answered by Jardim and Verbitsky in 2014) of the so-called **instanton component** of the moduli space $\mathcal{B}_{\mathbb{P}^3}(2, 0, c_2)$ for all $c_2 \in \mathbb{Z}_+$ remained open until 2014.

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In this talk, I'll present my joint paper with Ch. Almeida (Belo Horizonte), M. Jardim (Campinas), and Sergey Tikhomirov (Yaroslavl) [*New moduli components of rank 2 bundles on projective space*. Sbornik Mathematics, 212:11 (**2021**), 1503-1552.]

In this paper, we continue the study of the moduli space $\mathcal{B}_{\mathbb{P}^3}(2, 0, n)$, which we will simply denote by $\mathcal{B}(n)$ from now on, with the goal of providing new examples of families of vector bundles, and understanding their geometry. It is more or less clear from the table in [Hartshorne-Rao, 1991, Section 5.3] that $\mathcal{B}(1)$ and $\mathcal{B}(2)$ should be irreducible, while $\mathcal{B}(3)$ and $\mathcal{B}(4)$ should have exactly two irreducible components; see [Ellingsrud-Strømme, 1981] and [Chang, 1983], respectively, for the proof of the statements about $\mathcal{B}(3)$ and $\mathcal{B}(4)$.

As for $\mathcal{B}(5)$, a description of all its irreducible components had been a challenge since 1980ies. In the paper, we give a complete answer to this problem (Main Theorem 2 below).

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For $n \geq 5$, two families of irreducible components have been studied, namely the **instanton components**, and the **Ein components**, whose general point corresponds to a bundle given as cohomology of a monad of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-c) \to \mathcal{O}_{\mathbb{P}^3}(-b) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \to \mathcal{O}_{\mathbb{P}^3}(c) \to 0,$$

$$b \geq a \geq 0, c > a + b.$$

In 2019 A. Kytmanov, T, & S. Tikhomirov proved that the Ein components are **rational** varieties.

All of the components of $\mathcal{B}(n)$ for $n \leq 4$ are of either of these types; here we focus on a new family of bundles that appear as soon as $n \geq 5$. More precisely, we study the set of vector bundles in $\mathcal{B}(a^2 + k)$ for each $a \geq 2$ and $k \geq 1$ which arise as cohomologies of monads of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus k} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 2k+4} \to \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus k} \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0,$$

which will be denoted by $\mathcal{G}(a, k)$. We provide a bijection between such monads and monads of the form:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \to E \to \mathcal{O}_{\mathbb{P}^3}(a) \to 0,$$

where E is a symplectic rank 4 instanton bundle of charge k.

When $k = 1$, these facts are used to prove our first main result. (See Theorem 5.2 below.)

Main Theorem 1

For each $a \geq 2$ not equal to 3, $\mathcal{G}(a, 1)$ is a nonsingular dense subset of a rational *irreducible component of $\mathcal{B}(a^2 + 1)$ of dimension* $4\binom{a+3}{3} - a - 1$.

Our second main result provides a complete description of all the irreducible components of $\mathcal{B}(5)$.

Main Theorem 2

The moduli space $\mathcal{B}(5)$ has exactly 3 rational *irreducible components:*

(i) *the* instanton component, *of dimension 37, which is nonsingular and consists of those bundles given as cohomology of monads of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 5} \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 12} \to \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 5} \to 0, \qquad (1)$$

or of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 3} \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^3}(2)^{\oplus 2} \to 0; \qquad (2)$$

(ii) *the* Ein component, *nonsingular of dimension 40, which consists of those bundles given as cohomology of monads of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-3) \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to \mathcal{O}_{\mathbb{P}^3}(3) \to 0; \qquad (3)$$

(iii) *the closure of the set $\mathcal{G}(2,1)$, of dimension* 37 *consisting of the so-called* modified instantons *given as cohomology of monads of the form*

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2) \to 0 \qquad (4)$$

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Irreducible components of $\mathcal{B}(5)$

Component	**Dimension**	**Monads**	**Spectra**	**α-invariant**
Instanton	37	(1)	(0,0,0,0,0)	0
		(2)	(-1,-1,0,1,1)	
Ein	40	(3)	(-2,-1,0,1,2)	1
Modified Instanton	37	(4)	(-1,0,0,0,1)	1
		(5)		

Here α-invariant of a vector bundle E is $\alpha(E) := h^1(E(-2)) \bmod 2$.

Proof of Theorem 1

A vector bundle E is called **instanton bundle** if $h^i(E(-i-1)) = 0$, $i = 0, 1, 2, 3$. Here is a list of some properties of instanton bundles.

(i) Every rank 4 instanton bundle E of charge 1 satisfies an exact triple $0 \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 2} \to E \to N \to 0$, where N is a null-correlation bundle.

(ii) The cohomology bundle $E = \mathcal{H}^0(M^\bullet)$ of the monad $M^\bullet$ of the form:

$$M^\bullet : \quad 0 \to \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^3}(1) \to 0, \qquad (6)$$

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(iii) Any rank 2 bundle $[\mathcal{E}] \in \mathcal{G}(a, k)$ is the cohomology of a monad

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We construct three families of symplectic monads of the form (6). The first one is the universal family, with the base scheme S, of monads with E splitting as

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The second is a family, with the base scheme $\widetilde{S}$ containing S as a dense open subset, of monads with E a general *symplectic* rank 4 instanton of charge 1.

The third is a family of monads with E splitting as in the first one, but with a new base Y. All the three families inherit universal cohomology sheaves, and it is shown that the images of their corresponding modular morphisms to $\mathcal{B}(a^2+1)$ have the same closure $\overline{\mathcal{G}(a,1)}$.

We will give now more details of construction for this family Y. For this, introduce some new schemes and morphisms.

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We will give now more details of construction for this family Y. For this, introduce some new schemes and morphisms.

$B := \mathcal{B}(1)$, $\mathbb{B} := \mathbb{P}^3 \times B$, $\mathbb{N}$ universal family of bundles on $\mathbb{B}$,

$\mathbb{E} := \mathcal{O}_{\mathbb{B}} \oplus \mathbb{N}$, $\mathbb{P}^3_b := \mathbb{P}^3 \times \{b\}$, $E_b := \mathbb{E}|_{\mathbb{P}^3_b}$, $N_b := \mathbb{N}|_{\mathbb{P}^3_b}$, $b \in B$,

$\mathcal{T} := \{(b, \langle\sigma\rangle) \mid b \in B,\ 0 \neq \sigma \in H^0(E_b(a))\}$, $\mathcal{T} \to B$ projection,
$B_1 := \{(b, \varphi_1) \mid b \in B,\ \varphi_1 : \mathcal{O}_{\mathbb{P}^3_b}^{\oplus 2} \xrightarrow{\simeq} \mathcal{O}_{\mathbb{P}^3_b}^{\oplus 2}$ symplectic structure$\}$,
$B_2 := \{(b, \varphi_2) \mid b \in B,\ \varphi_2 : N_b \xrightarrow{\simeq} N_b^{\vee}$ symplectic structure$\}$,
$\tilde{B} := B_1 \times_B B_2$,

$Y := \tilde{B} \times_B \mathcal{T}$, $\mathbb{Y} := \mathbb{P}^3 \times Y$, $\mathbb{E}_{\mathbb{Y}} := \mathbb{E} \otimes_{\mathcal{O}_{\mathbb{B}}} \mathcal{O}_{\mathbb{Y}}$,
$L := \mathcal{O}_{Y/\tilde{B}}(1)$ Grothendieck sheaf, $\mathbb{P}^3_y := \mathbb{P}^3 \times \{y\}$, $y \in Y$.
Clearly, Y is a rational irreducible variety.

$\mathbb{A}^{\bullet}:\ 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \boxtimes L^{\vee} \to \mathbb{E}_{\mathbb{Y}} \to \mathcal{O}_{\mathbb{P}^3}(-a) \boxtimes L \to 0$ universal monad,
$\mathcal{E} := \mathcal{H}^0(\mathbb{A}^{\bullet})$ cohomology bundle of $\mathbb{A}^{\bullet}$

$\Phi_Y:\ Y \to \mathcal{B}(a^2+1)$, $y \mapsto [\mathcal{E}|_{\mathbb{P}^3_y}]$ modular morphism,
Similarly, there are well-defined modular morphisms
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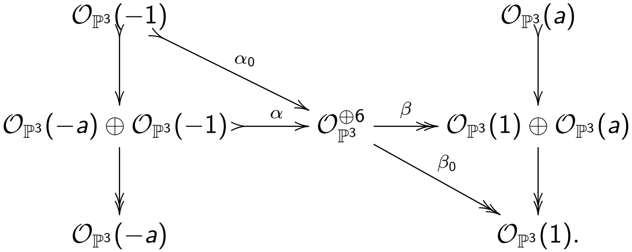
Proof of Theorem 1

Comments to the construction of S:
$\mathcal{G}(a,1) = \{[\mathcal{E}] \in \mathcal{B}(a^2+1) \mid \mathcal{E} = \mathcal{H}^0(A_S^\bullet)\}$, where $A_S^\bullet$ is a monad:

$$A_S^\bullet : \quad 0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(-1) \to \mathcal{O}_{\mathbb{P}^3}^{\oplus 6} \to \mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \to 0,$$



$E = \frac{\ker \beta_0}{\mathrm{im}\alpha_0}$:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-a) \to E \to \mathcal{O}_{\mathbb{P}^3}(a) \to 0 \qquad (7)$$

$\mathcal{E} = \mathcal{H}^0(\textit{monad } (7))$

Theorem

(i) $\Phi_{\tilde{S}}(\tilde{S}) = \mathcal{G}(a,1)$.
(ii) $\mathcal{G}(a,1)_0 := \Phi_Y(Y) = \Phi_S(S)$ *is a dense subset of* $\overline{\mathcal{G}(a,1)}$.
(iii) The modular morphism Φ_Y *factors as*

$$\Phi_Y : \ Y \xrightarrow{\pi} \mathcal{P} \hookrightarrow \mathcal{B}(a^2+1),$$

where $\mathcal{P}$ *is a* rational *variety and* $\pi : Y \to \mathcal{P}$ *is a principal* G*-bundle, where* $G \simeq GL(2,\mathrm{k}) \times \mathrm{k}^{\times}$*. Hence,* $\mathcal{P} = \mathcal{G}(a,1)_0$.
(iv) $\dim \mathcal{P} = 4\binom{a+3}{3} - a - 1 = h^1(\mathcal{E}nd(E_y))$ *for* $y \in Y$*. Hence,* $\overline{\mathcal{G}(a,1)}$ *is an irreducible component of* $\mathcal{B}(a^2+1)$.

The proof of this theorem is an explicit calculation, though quite involved, especially of statement (iii). Main Theorem 1 is a direct corollary of this theorem.

Proof of Theorem 2

Consider the set

$$\mathcal{H} = \{[\mathcal{E}] \in \mathcal{B}(5) \mid \mathcal{E} = \mathcal{H}(M^\bullet),\ \text{where } M^\bullet \text{ is a monad of type (5)}\},$$

$$M^\bullet:\ 0 \to M^{-1} \xrightarrow{\alpha} M^0 \xrightarrow{\beta} M^1 \to 0, \qquad M^1 = \mathcal{O}_{\mathbb{P}^3}(2) \oplus \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2},$$

$$M^0 = \mathcal{O}_{\mathbb{P}^3}(-1) \oplus V_6 \otimes \mathcal{O}_{\mathbb{P}^3}^{\oplus 6} \oplus \mathcal{O}_{\mathbb{P}^3}(1), \qquad M^{-1} = (M^1)^\vee.$$

It is known [Hartshorne-Rao, 1991, Table 5.3] that $\mathcal{H} \neq \emptyset$. Note that $\mathcal{H}$ is a constructible subset of $\mathcal{B}(5)$, as well as $\mathcal{G}(2,1)$. We prove

Theorem

$$\dim(\mathcal{H} \smallsetminus (\mathcal{G}(2,1) \cap \mathcal{H})) \leq 36.$$

Hence the closure of $\mathcal{H}$ in $\mathcal{B}(5)$ does not constitute a component of $\mathcal{B}(5)$.

The idea is to relate the vector bundle $[\mathcal{E}] \in \mathcal{H} \setminus (\mathcal{G}(2,1) \cap \mathcal{H})$ to a certain rank 2 reflexive sheaf

$$\mathcal{F} = \mathcal{F}(M^\bullet)$$

with Chern classes $c_1(\mathcal{F}) = 0$, $c_2(\mathcal{F}) = 2$ and $c_3(\mathcal{F}) = 2k$, $0 \leq k \leq 6$.

Namely, $M^\bullet$ yields a display diagram in which α_0 and β_0 are the induced morphisms:

$$\begin{array}{ccccc}
\mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2} & & & & \mathcal{O}_{\mathbb{P}^3}(2) \\
\downarrow & \searrow^{\alpha_0} & & & \downarrow \\
M^{-1} & \xrightarrow{\alpha} & M^0 & \xrightarrow{\beta} & M^1 \\
\downarrow & & & \searrow^{\beta_0} & \downarrow \\
\mathcal{O}_{\mathbb{P}^3}(-2) & & & & \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}.
\end{array} \qquad (1)$$

Since there is a unique (up to a scalar multiple) quotient morphism $M^0 \twoheadrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)$, we have well-defined morphisms

$$\tilde{\alpha}: \ \mathcal{O}_{\mathbb{P}^3}(-1)^{\oplus 2} \overset{\alpha_0}{\hookrightarrow} M^0 \twoheadrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)$$

$$\tilde{\beta}: \ \mathcal{O}_{\mathbb{P}^3}(1) \hookrightarrow M^0 \overset{\beta_0}{\twoheadrightarrow} \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 2}.$$

The sheaf $\mathcal{F}(M^\bullet)$ is constructed in the following way: It occurs that the only possible case for $\tilde{\alpha}$ and $\tilde{\beta}$ is

$$\tilde{\alpha} = \tilde{\beta} = 0.$$

This condition and some standard diagram chasing with the above display imply that there exist a uniquely defined monomorphism $j : \mathcal{O}_{\mathbb{P}^3}(1) \rightarrowtail E := \frac{\ker \beta_0}{\mathrm{im}\ \alpha_0}$ and, respectively, a uniquely defined epimorphism $\varepsilon : \mathrm{coker}(j) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)$. Then $\mathcal{F}(M^\bullet)$ is defined as

$$\mathcal{F}(M^\bullet) := \ker(\varepsilon).$$

Again, a diagram chasing with the above display induces a monad:

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-2) \xrightarrow{\sigma} E \to \mathcal{O}_{\mathbb{P}^3}(2) \to 0, \qquad \text{with} \qquad \mathcal{E} = \mathcal{H}^0(E),$$

and uniquely defined monomorphisms $j' : \mathcal{O}_{\mathbb{P}^3}(1) \rightarrowtail \mathrm{coker}(\sigma)$ and $j'' : \mathcal{O}_{\mathbb{P}^3}(-1) \rightarrowtail \mathcal{O}_{\mathbb{P}^3}$, and we set

$$\mathcal{L} = \mathcal{L}(M^\bullet) := \mathrm{coker}(j'), \qquad \mathbb{P}^2 = \mathbb{P}^2(M^\bullet) := \mathrm{Supp}(\mathrm{coker}(j'')).$$

Claim:

(i) The sheaf $\mathcal{L} = \mathcal{L}(M^\bullet)$ is a stable reflexive rank 2 sheaf on $\mathbb{P}^3$, $[\mathcal{L}] \in \mathcal{R}(1,4,6)$.
(ii) The sheaf $\mathcal{F} = \mathcal{F}(M^\bullet)$ is a reflexive rank 2 sheaf on $\mathbb{P}^3$, fitting in an exact triple

$$0 \to \mathcal{F} \to \mathcal{L} \to \mathcal{I}_{W,\mathbb{P}^2}(-1) \to 0,$$

and in its dual

$$0 \to \mathcal{L}(-1) \to \mathcal{F} \to \mathcal{I}_{Z,\mathbb{P}^2}(2) \to 0,$$

where $\mathbb{P}^2 = \mathbb{P}^2(M^\bullet)$, Z and W are subschemes of $\mathbb{P}^2$, $\dim Z \leq 0$, $\dim W \leq 0$, *and*

$$\ell(Z) + \ell(W) = 6.$$

Chern classes of $\mathcal{F}$ are $c_1(\mathcal{F}) = 0$, $c_2(\mathcal{F}) = 2$, $0 \leq c_3(\mathcal{F}) = 2\ell(W) \leq 12$,

i.e.,

$$[\mathcal{F}] \in \bigsqcup_{0 \leq k \leq 6} \mathcal{R}_k, \qquad \mathcal{R}_k := \mathcal{R}(0,2,2k).$$

The relation between the sheaf $\mathcal{E} = \mathcal{H}^0(M^\bullet)$ and the reflexive sheaf $\mathcal{F}$ constructed above is given by the following

Proposition

There is an inclusion

$$\mathcal{H} \smallsetminus (\mathcal{H} \cap \mathcal{G}(2,1)) \subset \bigsqcup_{0 \le k \le 6} \mathcal{H}_k, \qquad \textit{where}$$

$$\mathcal{H}_k = \{[\mathcal{E}] \in \mathcal{B}(5) \mid \mathcal{E} \text{ is obtained from } \mathcal{F}, \text{where } [\mathcal{F}] \in \mathcal{R}_k,$$
$$\text{by the two subsequent elementary transformations (1) below}\},$$

$$0 \to \mathcal{L}(-1) \to \mathcal{F} \to \mathcal{I}_{Z,\mathbb{P}^2}(2) \to 0, \qquad (\textit{step } 1)$$
$$0 \to \mathcal{E} \to \mathcal{L} \to \mathcal{O}_{\mathbb{P}^2}(2) \to 0, \qquad (\textit{step } 2)$$

where $\mathbb{P}^2$ is some plane in $\mathbb{P}^3$, $Z \subset \mathbb{P}^2$, $\dim Z \le 0$, $\ell(Z) = 6 - k$, and $\mathcal{L}$ is a stable reflexive sheaf from $\mathcal{R}(1,4,6)$.

Properties of the reflexive sheaf $\mathcal{F}$ are reflected in the following statements. (Here we denote by $\mathcal{R}_k^s$ and $\mathcal{R}_k^u$ the moduli spaces of stable and unstable reflexive sheaves from $\mathcal{R}_k$, respectively.)

Claim:

(i) $\mathcal{R}_k^u \neq \emptyset$ only for $0 \leq k \leq 3$, and any sheaf $\mathcal{F}$ from $\mathcal{R}_k^u$ fits in an exact triple

$$0 \to \mathcal{O}_{\mathbb{P}^3} \xrightarrow{s} \mathcal{F} \xrightarrow{u} \mathcal{I}_{C,\mathbb{P}^3} \to 0,$$

where $C = \mathrm{Sing}(\mathcal{F}/\mathcal{O}_{\mathbb{P}^3})$ is a l.c.i. curve of degree 2 in $\mathbb{P}^3$, $\chi(\mathcal{O}_C) = 4 - \frac{1}{2}c_3(\mathcal{F}) = 4 - k$.
(ii) If C is reduced, then either $c_3(\mathcal{F}) = 4$ and C is a disjoint union $l_1 \sqcup l_2$ of two projective lines in $\mathbb{P}^3$, or $c_3(\mathcal{F}) = 6$, then C is a plane conic in $\mathbb{P}^3$.
(iii) If C is nonreduced then C is the scheme structure of multiplicity two on a projective line l in $\mathbb{P}^3$ defined by an exact sequence

$$0 \to \mathcal{I}_{C,\mathbb{P}^3} \to \mathcal{I}_{l,\mathbb{P}^3} \to \mathcal{O}_l(m) \to 0, \qquad -1 \leq m = 2 - k \leq 2.$$

(iv) The moduli spaces $\mathcal{R}_k^u$ are varieties of dimensions $\dim \mathcal{R}_0^u = \dim \mathcal{R}_3^u = 14$, $\dim \mathcal{R}_1^u = \dim \mathcal{R}_2^u = 13$, and they are fine.

Claim:

Suppose that $[\mathcal{F}] \in \mathcal{R}_k^s$. Then the following statements hold.
(i) $\mathcal{R}_k^s \neq \emptyset$ only for $0 \leq k \leq 2$.
(ii) $\dim \mathcal{R}_k^s = 13, \ k = 0, 1, 2$.
(iii) For $0 \leq k \leq 2$ and any $[\mathcal{F}] \in \mathcal{R}_k^s$,
$\dim \mathrm{Ext}^1(\mathcal{F}, \mathcal{F}) = 13, \quad \mathrm{Ext}^2(\mathcal{F}, \mathcal{F}) = 0$.
(iv) For any $\mathbb{P}^2 \subset \mathbb{P}^3$, $h^0(\mathcal{F}_{\mathbb{P}^2}(2)) = 10$, $h^1(\mathcal{F}_{\mathbb{P}^2}(2)) = 0$.

Using these two claims, together with the above Proposition on a pair of elementary transformations from $\mathcal{F}$ to $\mathcal{E}$, we eventually obtain the desired result that $\dim(\mathcal{H} \setminus (\mathcal{G}(2,1) \cap \mathcal{H})) \leq 36$.

To finish the proof of Theorem 2, we make the following remarks.

The first ingredient is the result of [Hartshorne-Rao, 1991, Table 5.3, case 5.(1)-(4)] saying that every bundle in $\mathcal{B}(5)$ is cohomology of one of the monads (1)-(5).

Claim:

Suppose that $[\mathcal{F}] \in \mathcal{R}_k^s$. Then the following statements hold.
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(ii) $\dim \mathcal{R}_k^s = 13,\ k = 0, 1, 2$.
(iii) For $0 \leq k \leq 2$ *and any* $[\mathcal{F}] \in \mathcal{R}_k^s$,
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It is known that the Atiyah-Rees α-invariant of E is invariant on the connected components of the moduli space of stable vector bundles on $\mathbb{P}^3$. One can easily check that the cohomologies of monads of the form (1) and (2) have α-invariant equal to 0, while the cohomologies of the monads (3), (4) and (5) have α-invariant equal to 1.

Rao, 1987: the family of cohomology bundles of monads of the form (2) is irreducible, of dimension 36, and it lies in a *unique* component of $\mathcal{B}(5)$. Since instanton bundles of charge 5, i. e. the cohomologies of monads (1), yield an irreducible family of dimension 37, it follows that the set

$$\mathcal{I} := \{[E] \in \mathcal{B}(5) \mid \alpha(E) = 0\} \qquad (*)$$

forms a single irreducible component of $\mathcal{B}(5)$, of dimension 37, whose generic point corresponds to an instanton bundle. In addition, every $[E] \in \mathcal{I}$ satisfies $h^1(\mathcal{E}nd(E)) = 37$; this was originaly proved by Katsylo and Ottaviani in 2004 for instanton bundles, and by Rao in 1987 for the cohomologies of monads (2). Therefore, $\mathcal{I}$ is nonsingular. This completes the proof of the first statement (i) of the Main Theorem 1.

It is known that the Atiyah-Rees α-invariant of E is invariant on the connected components of the moduli space of stable vector bundles on $\mathbb{P}^3$. One can easily check that the cohomologies of monads of the form (1) and (2) have α-invariant equal to 0, while the cohomologies of the monads (3), (4) and (5) have α-invariant equal to 1.

Rao, 1987: the family of cohomology bundles of monads of the form (2) is irreducible, of dimension 36, and it lies in a *unique* component of $\mathcal{B}(5)$. Since instanton bundles of charge 5, i. e. the cohomologies of monads (1), yield an irreducible family of dimension 37, it follows that the set

$$\mathcal{I} := \{[E] \in \mathcal{B}(5) \mid \alpha(E) = 0\} \qquad (*)$$

forms a single irreducible component of $\mathcal{B}(5)$, of dimension 37, whose generic point corresponds to an instanton bundle. In addition, every $[E] \in \mathcal{I}$ satisfies $h^1(\mathcal{E}nd(E)) = 37$; this was originaly proved by Katsylo and Ottaviani in 2004 for instanton bundles, and by Rao in 1987 for the cohomologies of monads (2). Therefore, $\mathcal{I}$ is nonsingular. This completes the proof of the first statement (i) of the Main Theorem 1.

Our next step is to analyse bundles with α-invariant equal to 1. Hartshorne, 1980: the family $\mathcal{K}$ of stable rank 2 bundles E with $c_1(E) = 0$ and $c_2(E) = 5$ with spectrum $(-2, -1, 0, 1, 2)$ is an irreducible, nonsigular family of dimension 40, and from the definition of spectrum one has

$$h^1(\mathcal{E}(-2)) = 3, \qquad [\mathcal{E}] \in \mathcal{K}. \qquad (**)$$

[Hartshorne-Rao, 1991, Table 5.3, case 5.(4)]: bundles from $\mathcal{K}$ are precisely those given as cohomologies of monads (3). This is a particular case of a class of monads studied by Ein in 1988. Ein shows that the closure $\overline{\mathcal{K}}$ of $\mathcal{K}$ in $\mathcal{B}(5)$ is an *irreducible component* of $\mathcal{B}(5)$ of dimension 40.

Main Theorem 1, case $a = 2$: bundles arising as cohomology of monads (4) (*modified instantons*) form a dense subset $\mathcal{G}(2,1)$ of a *rational irreducible component* of dimension 37. Consider the above studied set $\mathcal{H}$ of cohomology bundles of monads (5). Since the bundles from $\mathcal{G}(2,1) \cup \mathcal{H}$ have the spectrum $(-1, 0, 0, 0, 1)$ by [Hartshorne-Rao, 1991, Table 5.3, case 5.(2)], we have

$$h^1(\mathcal{E}(-2)) = 1, \qquad [\mathcal{E}] \in \mathcal{G}(2,1) \cup \mathcal{H}, \qquad (***)$$

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Main Theorem 1, case $a = 2$: bundles arising as cohomology of monads (4) (*modified instantons*) form a dense subset $\mathcal{G}(2, 1)$ of a *rational irreducible component* of dimension 37. Consider the above studied set $\mathcal{H}$ of cohomology bundles of monads (5). Since the bundles from $\mathcal{G}(2, 1) \cup \mathcal{H}$ have the spectrum $(-1, 0, 0, 0, 1)$ by [Hartshorne-Rao, 1991, Table 5.3, case 5.(2)], we have

$$h^1(\mathcal{E}(-2)) = 1, \qquad [\mathcal{E}] \in \mathcal{G}(2, 1) \cup \mathcal{H}, \qquad (***)$$

so that $\alpha(\mathcal{E}) = 1$, and therefore, in view of (*), $\mathcal{H} \cap \mathcal{I} = \emptyset$. As we have seen in Theorem on the dimension of $\mathcal{H}$, $\mathcal{H}$ does not constitute a component in $\mathcal{B}(5)$, it then follows from the above that

$$\mathcal{H} \subset \overline{\mathcal{G}(2,1)} \cup \overline{\mathcal{K}}.$$

Proposition

$\mathcal{H} \subset \overline{\mathcal{G}(2,1)}$ and $\overline{\mathcal{K}} = \mathcal{K}$.

Proof. We only have to show that $(\mathcal{G}(2,1) \cup \mathcal{H}) \cap \overline{\mathcal{K}} = \emptyset$. Suppose by contradiction that there exists a vector bundle $[\mathcal{E}] \in (\mathcal{G}(2,1) \cup \mathcal{H}) \cap \overline{\mathcal{K}}$. By (**) and the inferior semi-continuity of the dimension of the cohomology groups of coherent sheaves, one has that $h^1(\mathcal{E}(-2)) \geq 3$, contrary to (***). □

This last proposition finally concludes the proof of Main Theorem 2.